Solución al problema de "The integral equals"

Enunciado:

The integral $\int_0^{\pi} \sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals: ?

Solución:

$$1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}=\left(1-2\sin\frac{x}{2}\right)^2 \Leftrightarrow \sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}=\left|1-2\sin\frac{x}{2}\right|$$

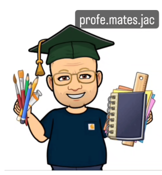
Y nuestra integral queda: $\int_0^{\pi} \sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx=\int_0^{\pi}\left|1-2\sin\frac{x}{2}\right| dx$

Veamos el signo de $\left|1-2\sin\frac{x}{2}\right|$ en el intervalo $[0,\pi]$.

Llamemos $g(x)=1-2\sin\frac{x}{2}$, $x\in[0,\pi]$.

$g(x)=1-2\sin\frac{x}{2}=0 \Leftrightarrow \sin\frac{x}{2}=\frac{1}{2} \Rightarrow x=\frac{\pi}{3}$ en el intervalo $[0,\pi]$; al tratarse de una función continua (***g***) ($g\left(\frac{\pi}{3}\right)=0$, ***g(0)>0*** y ***g(π)<0***), ***g*** es positiva en $\left(0,\frac{\pi}{3}\right)$ y negativa en $\left(\frac{\pi}{3},0\right)$.

Luego: $f(x)=|g(x)|=\left|1-2\sin\frac{x}{2}\right|=\begin{cases} 1-2\sin\frac{x}{2} & si\ x\in\left[0,\frac{\pi}{3}\right) \\ 2\sin\frac{x}{2}-1 & si\ x\in\left[\frac{\pi}{3},\pi\right] \end{cases}$

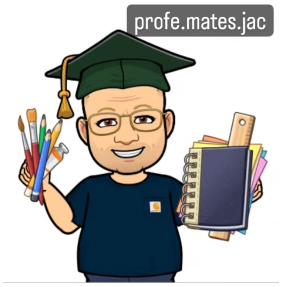


Por tanto:

$$\int_{0}^{\pi}\left|1-2\sin\frac{x}{2}\right|dx=\int_{0}^{\pi/3}1-2\sin\frac{x}{2}\,dx+\int_{\pi/3}^{\pi}2\sin\frac{x}{2}-1\,dx=\left(x+4\cos\frac{x}{2}\right)\Bigg|_{0}^{\frac{\pi}{3}}+\left(-x-4\cos\frac{x}{2}\right)\Bigg|_{\frac{\pi}{3}}^{\pi}=$$

$$=\frac{\pi}{3}+4\cos\frac{\pi}{6}-4\cos 0-\pi-4\cos\frac{\pi}{2}+\frac{\pi}{3}+4\cos\frac{\pi}{6}=\frac{\pi}{3}+2\sqrt{3}-4-\pi+\frac{\pi}{3}+2\sqrt{3}=$$

$$=4\sqrt{3}-\frac{\pi}{3}-4$$



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